

Contents

1	Differential manifolds	1
1.A	From submanifolds to abstract manifolds	2
1.A.1	Submanifolds of Euclidean spaces	2
1.A.2	Abstract manifolds	6
1.A.3	Smooth maps	10
1.B	The tangent bundle	12
1.B.1	Tangent space to a submanifold of $\mathbf{R}^{n+k}$	12
1.B.2	The manifold of tangent vectors	14
1.B.3	Vector bundles	16
1.B.4	Tangent map	17
1.C	Vector fields	17
1.C.1	Definitions	17
1.C.2	Another definition for the tangent space	19
1.C.3	Integral curves and flow of a vector field	22
1.C.4	Image of a vector field by a diffeomorphism	24
1.D	Baby Lie groups	26
1.D.1	Definitions	26
1.D.2	Adjoint representation	28
1.E	Covering maps and fibrations	29
1.E.1	Covering maps and quotients by a discrete group	29
1.E.2	Submersions and fibrations	31
1.E.3	Homogeneous spaces	32
1.F	Tensors	35
1.F.1	Tensor product (a digest)	35
1.F.2	Tensor bundles	36
1.F.3	Operations on tensors	37
1.F.4	Lie derivatives	38
1.F.5	Local operators, differential operators	39
1.F.6	A characterization for tensors	40
1.G	Differential forms	41
1.G.1	Definitions	41

1.G.2	Exterior derivative	42
1.G.3	Volume forms	45
1.G.4	Integration on an oriented manifold	46
1.G.5	Haar measure on a Lie group	47
1.H	Partitions of unity	47
2	Riemannian metrics	51
2.A	Existence theorems and first examples	52
2.A.1	Basic definitions	52
2.A.2	Submanifolds of Euclidean or Minkowski spaces	54
2.A.3	Riemannian submanifolds, Riemannian products	58
2.A.4	Riemannian covering maps, flat tori	59
2.A.5	Riemannian submersions, complex projective space	63
2.A.6	Homogeneous Riemannian spaces	65
2.B	Covariant derivative	70
2.B.1	Connections	70
2.B.2	Canonical connection of a Riemannian submanifold	72
2.B.3	Extension of the covariant derivative to tensors	73
2.B.4	Covariant derivative along a curve	75
2.B.5	Parallel transport	78
2.B.6	A natural metric on the tangent bundle	80
2.C	Geodesics	80
2.C.1	Definition, first examples	80
2.C.2	Local existence and uniqueness for geodesics, exponential map	85
2.C.3	Riemannian manifolds as metric spaces	89
2.C.4	An invitation to isosystolic inequalities	94
2.C.5	Complete Riemannian manifolds, Hopf-Rinow theorem	96
2.C.6	Geodesics and submersions, geodesics of $P^n\mathbf{C}$:	100
2.C.7	Cut-locus	103
2.C.8	The geodesic flow	109
2.D	A glance at pseudo-Riemannian manifolds	115
2.D.1	What remains true?	115
2.D.2	Space, time and light-like curves	116
2.D.3	Lorentzian analogs of Euclidean spaces, spheres and hyperbolic spaces	117
2.D.4	(In)completeness	120
2.D.5	The Schwarzschild model	121
2.D.6	Hyperbolicity versus ellipticity	126
3	Curvature	129
3.A	The curvature tensor	130
3.A.1	Second covariant derivative	130
3.A.2	Algebraic properties of the curvature tensor	131
3.A.3	Computation of curvature: some examples	133
3.A.4	Ricci curvature, scalar curvature	135

3.B	First and second variation	136
3.B.1	Technical preliminaries	137
3.B.2	First variation formula	138
3.B.3	Second variation formula	139
3.C	Jacobi vector fields	141
3.C.1	Basic topics about second derivatives	141
3.C.2	Index form	142
3.C.3	Jacobi fields and exponential map	145
3.C.4	Applications	146
3.D	Riemannian submersions and curvature	148
3.D.1	Riemannian submersions and connections	148
3.D.2	Jacobi fields of $P^n\mathbf{C}$	149
3.D.3	O'Neill's formula	151
3.D.4	Curvature and length of small circles. Application to Riemannian submersions	152
3.E	The behavior of length and energy in the neighborhood of a geodesic	154
3.E.1	Gauss lemma	154
3.E.2	Conjugate points	155
3.E.3	Some properties of the cut-locus	159
3.F	Manifolds with constant sectional curvature	160
3.G	Topology and curvature: two basic results	162
3.G.1	Myers' theorem	162
3.G.2	Cartan-Hadamard's theorem	163
3.H	Curvature and volume	164
3.H.1	Densities on a differentiable manifold	164
3.H.2	Canonical measure of a Riemannian manifold	165
3.H.3	Examples: spheres, hyperbolic spaces, complex projective spaces	167
3.H.4	Small balls and scalar curvature	168
3.H.5	Volume estimates	169
3.I	Curvature and growth of the fundamental group	173
3.I.1	Growth of finite type groups	173
3.I.2	Growth of the fundamental group of compact manifolds with negative curvature	174
3.J	Curvature and topology: some important results	176
3.J.1	Integral formulas	176
3.J.2	(Geo)metric methods	177
3.J.3	Analytic methods	178
3.J.4	Coarse point of view: compactness theorems	179
3.K	Curvature tensors and representations of the orthogonal group	180
3.K.1	Decomposition of the space of curvature tensors	180
3.K.2	Conformally flat manifolds	183
3.K.3	The Second Bianchi identity	184

3.L	Hyperbolic geometry	185
3.L.1	Introduction	185
3.L.2	Angles and distances in the hyperbolic plane	185
3.L.3	Polygons with "many" right angles	190
3.L.4	Compact surfaces	193
3.L.5	Hyperbolic trigonometry	194
3.L.6	Prescribing constant negative curvature	198
3.L.7	A few words about higher dimension	200
3.M	Conformal geometry	201
3.M.1	Introduction	201
3.M.2	The Möbius group	201
3.M.3	Conformal, elliptic and hyperbolic geometry	204
4	Analysis on manifolds	207
4.A	Manifolds with boundary	207
4.A.1	Definition	207
4.A.2	Stokes theorem and integration by parts	209
4.B	Bishop inequality	212
4.B.1	Some commutation formulas	212
4.B.2	Laplacian of the distance function	213
4.B.3	Another proof of Bishop's inequality	214
4.B.4	Heintze-Karcher inequality	216
4.C	Differential forms and cohomology	217
4.C.1	The de Rham complex	217
4.C.2	Differential operators and their formal adjoints	218
4.C.3	The Hodge-de Rham theorem.	220
4.C.4	A second visit to the Bochner method	221
4.D	Basic spectral geometry	223
4.D.1	The Laplace operator and the wave equation	223
4.D.2	Statement of basic results on the spectrum	225
4.E	Some examples of spectra	227
4.E.1	Introduction	227
4.E.2	The spectrum of flat tori	227
4.E.3	Spectrum of (S^n, can)	228
4.F	The minimax principle	231
4.G	Eigenvalues estimates	235
4.G.1	Introduction	235
4.G.2	Bishop's inequality and coarse estimates	235
4.G.3	Some consequences of Bishop's theorem	235
4.G.4	Lower bounds for the first eigenvalue	238
4.H	Paul Levy's isoperimetric inequality	240
4.H.1	The statement	240
4.H.2	The proof	241

5	Riemannian submanifolds	245
5.A	Curvature of submanifolds	245
5.A.1	Second fundamental form	245
5.A.2	Curvature of hypersurfaces	248
5.A.3	Application to explicit computations of curvatures	250
5.B	Curvature and convexity	253
5.C	Minimal surfaces	257
5.C.1	First results	257
5.C.2	Surfaces with constant mean curvature	261
A	Some extra problems	263
B	Solutions of exercises	265
	Bibliography	305
	Index	315
	List of figures	321