



Tessellations: Mathematics, Art, and Recreation aims to present a comprehensive introduction to tessellations (tiling) at a level accessible to non-specialists. Additionally, it covers techniques, tips, and templates to facilitate the creation of mathematical art based on tessellations. Inclusion of special topics like spiral tilings and tessellation metamorphoses allows the reader to explore beautiful and entertaining math and art.

The book has a particular focus on “Escheresque” designs, in which the individual tiles are recognizable real-world motifs. These are extremely popular with students and math hobbyists but are typically very challenging to execute. Techniques demonstrated in the book are aimed at making these designs more achievable. Going beyond planar designs, the book contains numerous nets of polyhedra and templates for applying Escheresque designs to them.

Activities and worksheets are spread throughout the book, and examples of real-world tessellations are also provided.

Key features

- Introduces the mathematics of tessellations, including symmetry
- Covers polygonal, aperiodic, and non-Euclidean tilings
- Contains tutorial content on designing and drawing Escheresque tessellations
- Highlights numerous examples of tessellations in the real world
- Provides activities for individuals or classes
- Is filled with templates to aid in creating Escheresque tessellations
- Treats special topics like tiling rosettes, fractal tessellations, and decoration of tiles

Robert Fathauer has had a life-long interest in art but studied physics and mathematics in college, going on to earn a PhD from Cornell University in electrical engineering. For several years he was a researcher at the Jet Propulsion Laboratory in Pasadena, California. Long a fan of M.C. Escher, he began designing his own tessellations with lifelike motifs in the late 1980s. In 1993, he founded a business, Tessellations, to produce puzzles based on his designs. Over time, Tessellations has grown to include mathematics manipulatives, polyhedral dice, and books.

Dr. Fathauer’s mathematical art has always been coupled with recreational math explorations. These include Escheresque tessellations, fractal tilings, and iterated knots. After many years of creating two-dimensional art, he has recently been building ceramic sculptures inspired by both mathematics and biological forms. Another interest of his is photographing mathematics in natural and synthetic objects, particularly tessellations. In addition to creating mathematical art, he’s strongly committed to promoting it through group exhibitions at both the Bridges Conference and the Joint Mathematics Meetings.



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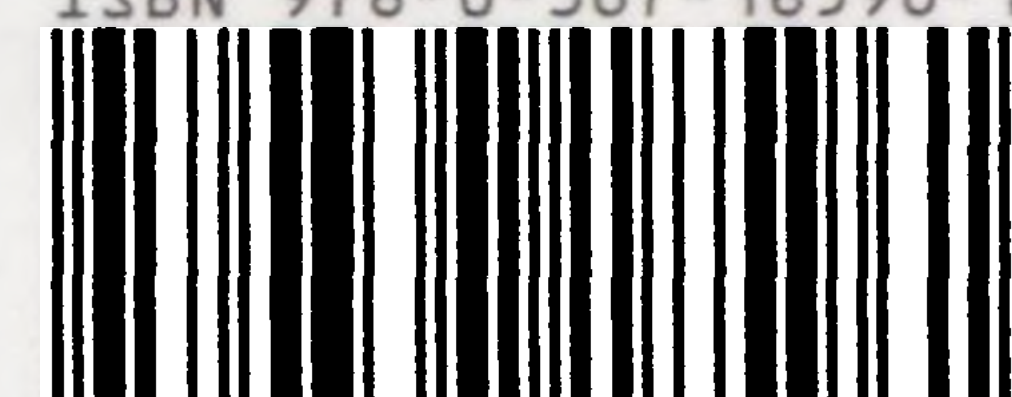
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