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In this chapter we discuss two mathematical formulations of the intuitive notion of a curve. The precise relation between them turns out to be quite subtle, so we shall begin by giving some examples of curves of each type and practical ways of passing between them.

1.1. What is a Curve?

If asked to give an example of a curve, you might give a straight line, say $y - 2x = 1$ (even though this is not 'curved'!), or a circle, say $x^2 + y^2 = 1$, or perhaps a parabola, say $y - x^2 = 0$.



All of these curves are described by means of their cartesian equation $f(x,y) = c$, where f is a function of x and y and c is a constant. From this point of view,