

CONTENTS

Preface	xv
1. Introduction	1
1.1. Partial differential equations	1
1.2. Examples	3
1.2.1. Single partial differential equations	3
1.2.2. Systems of partial differential equations	6
1.3. Strategies for studying PDE	7
1.3.1. Well-posed problems, classical solutions	7
1.3.2. Weak solutions and regularity	7
1.3.3. Typical difficulties	9
1.4. Overview	9
1.5. Problems	12

PART I: REPRESENTATION FORMULAS FOR SOLUTIONS

2. Four Important Linear PDE	17
2.1. Transport equation	18
2.1.1. Initial-value problem	18
2.1.2. Nonhomogeneous problem	19
2.2. Laplace's equation	20
2.2.1. Fundamental solution	21
2.2.2. Mean-value formulas	25

2.2.3. Properties of harmonic functions	27
2.2.4. Green's function	33
2.2.5. Energy methods	41
2.3. Heat equation	44
2.3.1. Fundamental solution	45
2.3.2. Mean-value formula	51
2.3.3. Properties of solutions	54
2.3.4. Energy methods	62
2.4. Wave equation	65
2.4.1. Solution by spherical means	67
2.4.2. Nonhomogeneous problem	81
2.4.3. Energy methods	83
2.5. Problems	85
2.6. References	89
3. Nonlinear First-Order PDE	91
3.1. Complete integrals, envelopes	92
3.1.1. Complete integrals	92
3.1.2. New solutions from envelopes	94
3.2. Characteristics	97
3.2.1. Derivation of characteristic ODE	97
3.2.2. Examples	99
3.2.3. Boundary conditions	103
3.2.4. Local solution	106
3.2.5. Applications	110
3.3. Introduction to Hamilton–Jacobi equations	115
3.3.1. Calculus of variations, Hamilton's ODE	116
3.3.2. Legendre transform, Hopf–Lax formula	121
3.3.3. Weak solutions, uniqueness	129
3.4. Introduction to conservation laws	136
3.4.1. Shocks, entropy condition	137
3.4.2. Lax–Oleinik formula	144
3.4.3. Weak solutions, uniqueness	149
3.4.4. Riemann's problem	154
3.4.5. Long time behavior	157

3.5. Problems 162

3.6. References 165

4. Other Ways to Represent Solutions 167

4.1. Separation of variables 167

4.2. Similarity solutions 172

4.2.1. Plane and traveling waves, solitons 172

4.2.2. Similarity under scaling 180

4.3. Transform methods 182

4.3.1. Fourier transform 182

4.3.2. Laplace transform 191

4.4. Converting nonlinear into linear PDE 194

4.4.1. Hopf–Cole transformation 194

4.4.2. Potential functions 196

4.4.3. Hodograph and Legendre transforms 197

4.5. Asymptotics 199

4.5.1. Singular perturbations 199

4.5.2. Laplace’s method 204

4.5.3. Geometric optics, stationary phase 206

4.5.4. Homogenization 218

4.6. Power series 221

4.6.1. Noncharacteristic surfaces 221

4.6.2. Real analytic functions 226

4.6.3. Cauchy–Kovalevskaya Theorem 228

4.7. Problems 233

4.8. References 235

**PART II: THEORY FOR LINEAR PARTIAL
DIFFERENTIAL EQUATIONS**

5. Sobolev Spaces 239

5.1. Hölder spaces 240

5.2. Sobolev spaces 241

5.2.1. Weak derivatives 242

5.2.2. Definition of Sobolev spaces 244

5.2.3. Elementary properties 247

5.3. Approximation 250

5.3.1. Interior approximation by smooth functions . . .	250
5.3.2. Approximation by smooth functions	251
5.3.3. Global approximation by smooth functions	252
5.4. Extensions	254
5.5. Traces	257
5.6. Sobolev inequalities	261
5.6.1. Gagliardo–Nirenberg–Sobolev inequality	262
5.6.2. Morrey’s inequality	266
5.6.3. General Sobolev inequalities	269
5.7. Compactness	271
5.8. Additional topics	275
5.8.1. Poincaré’s inequalities	275
5.8.2. Difference quotients	277
5.8.3. Differentiability a.e.	280
5.8.4. Fourier transform methods	282
5.9. Other spaces of functions	283
5.9.1. The space H^{-1}	283
5.9.2. Spaces involving time	285
5.10. Problems	289
5.11. References	292
6. Second-Order Elliptic Equations	293
6.1. Definitions	293
6.1.1. Elliptic equations	293
6.1.2. Weak solutions	295
6.2. Existence of weak solutions	297
6.2.1. Lax–Milgram Theorem	297
6.2.2. Energy estimates	299
6.2.3. Fredholm alternative	302
6.3. Regularity	308
6.3.1. Interior regularity	309
6.3.2. Boundary regularity	316
6.4. Maximum principles	326
6.4.1. Weak maximum principle	327
6.4.2. Strong maximum principle	330

6.4.3. Harnack's inequality 333

6.5. Eigenvalues and eigenfunctions 334

6.5.1. Eigenvalues of symmetric elliptic operators 334

6.5.2. Eigenvalues of nonsymmetric elliptic operators 340

6.6. Problems 345

6.7. References 347

7. Linear Evolution Equations 349

7.1. Second-order parabolic equations 349

7.1.1. Definitions 350

7.1.2. Existence of weak solutions 353

7.1.3. Regularity 358

7.1.4. Maximum principles 367

7.2. Second-order hyperbolic equations 377

7.2.1. Definitions 377

7.2.2. Existence of weak solutions 380

7.2.3. Regularity 387

7.2.4. Propagation of disturbances 394

7.2.5. Equations in two variables 397

7.3. Hyperbolic systems of first-order equations 400

7.3.1. Definitions 400

7.3.2. Symmetric hyperbolic systems 402

7.3.3. Systems with constant coefficients 408

7.4. Semigroup theory 412

7.4.1. Definitions, elementary properties 413

7.4.2. Generating contraction semigroups 418

7.4.3. Applications 420

7.5. Problems 425

7.6. References 427

**PART III: THEORY FOR NONLINEAR PARTIAL
DIFFERENTIAL EQUATIONS**

8. The Calculus of Variations 431

8.1. Introduction 431

8.1.1. Basic ideas 431

8.1.2. First variation, Euler–Lagrange equation 432

8.1.3. Second variation	436
8.1.4. Systems	437
8.2. Existence of minimizers	443
8.2.1. Coercivity, lower semicontinuity	443
8.2.2. Convexity	445
8.2.3. Weak solutions of Euler–Lagrange equation	450
8.2.4. Systems	453
8.3. Regularity	458
8.3.1. Second derivative estimates	458
8.3.2. Remarks on higher regularity	461
8.4. Constraints	463
8.4.1. Nonlinear eigenvalue problems	463
8.4.2. Unilateral constraints, variational inequalities	467
8.4.3. Harmonic maps	470
8.4.4. Incompressibility	472
8.5. Critical points	476
8.5.1. Mountain Pass Theorem	476
8.5.2. Application to semilinear elliptic PDE	482
8.6. Problems	486
8.7. References	489
9. Nonvariational Techniques	491
9.1. Monotonicity methods	491
9.2. Fixed point methods	498
9.2.1. Banach’s Fixed Point Theorem	498
9.2.2. Schauder’s, Schaefer’s Fixed Point Theorems	502
9.3. Method of subsolutions and supersolutions	507
9.4. Nonexistence	511
9.4.1. Blow-up	511
9.4.2. Derrick–Pohozaev identity	514
9.5. Geometric properties of solutions	517
9.5.1. Star-shaped level sets	517
9.5.2. Radial symmetry	518
9.6. Gradient flows	523
9.6.1. Convex functions on Hilbert spaces	523

9.6.2. Subdifferentials and nonlinear semigroups	528
9.6.3. Applications	534
9.7. Problems	536
9.8. References	538
10. Hamilton–Jacobi Equations	539
10.1. Introduction, viscosity solutions	539
10.1.1. Definitions	541
10.1.2. Consistency	543
10.2. Uniqueness	546
10.3. Control theory, dynamic programming	550
10.3.1. Introduction to control theory	551
10.3.2. Dynamic programming	552
10.3.3. Hamilton–Jacobi–Bellman equation	554
10.3.4. Hopf–Lax formula revisited	560
10.4. Problems	563
10.5. References	564
11. Systems of Conservation Laws	567
11.1. Introduction	567
11.1.1. Integral solutions	570
11.1.2. Traveling waves, hyperbolic systems	572
11.2. Riemann’s problem	579
11.2.1. Simple waves	579
11.2.2. Rarefaction waves	582
11.2.3. Shock waves, contact discontinuities	583
11.2.4. Local solution of Riemann’s problem	590
11.3. Systems of two conservation laws	592
11.3.1. Riemann invariants	593
11.3.2. Nonexistence of smooth solutions	597
11.4. Entropy criteria	599
11.4.1. Vanishing viscosity, traveling waves	600
11.4.2. Entropy/entropy-flux pairs	604
11.4.3. Uniqueness for a scalar conservation law	606
11.5. Problems	611
11.6. References	612

APPENDICES

Appendix A: Notation	613
A.1. Notation for matrices	613
A.2. Geometric notation	614
A.3. Notation for functions	615
A.4. Vector-valued functions	619
A.5. Notation for estimates	619
A.6. Some comments about notation	620
Appendix B: Inequalities	621
B.1. Convex functions	621
B.2. Elementary inequalities	622
Appendix C: Calculus Facts	626
C.1. Boundaries	626
C.2. Gauss–Green Theorem	627
C.3. Polar coordinates, coarea formula	628
C.4. Convolution and smoothing	629
C.5. Inverse Function Theorem	632
C.6. Implicit Function Theorem	633
C.7. Uniform convergence	634
Appendix D: Linear Functional Analysis	635
D.1. Banach spaces	635
D.2. Hilbert spaces	636
D.3. Bounded linear operators	637
D.4. Weak convergence	639
D.5. Compact operators, Fredholm theory	640
D.6. Symmetric operators	644
Appendix E: Measure Theory	645
E.1. Lebesgue measure	645
E.2. Measurable functions and integration	647
E.3. Convergence theorems for integrals	648
E.4. Differentiation	648
E.5. Banach space-valued functions	649
Bibliography	651
Index	655