

Contents

Preface	xii
Part 1. Large graphs: an informal introduction	1
Chapter 1. Very large networks	3
1.1. Huge networks everywhere	3
1.2. What to ask about them?	4
1.3. How to obtain information about them?	5
1.4. How to model them?	8
1.5. How to approximate them?	11
1.6. How to run algorithms on them?	18
1.7. Bounded degree graphs	22
Chapter 2. Large graphs in mathematics and physics	25
2.1. Extremal graph theory	25
2.2. Statistical physics	32
Part 2. The algebra of graph homomorphisms	35
Chapter 3. Notation and terminology	37
3.1. Basic notation	37
3.2. Graph theory	38
3.3. Operations on graphs	39
Chapter 4. Graph parameters and connection matrices	41
4.1. Graph parameters and graph properties	41
4.2. Connection matrices	42
4.3. Finite connection rank	45
Chapter 5. Graph homomorphisms	55
5.1. Existence of homomorphisms	55
5.2. Homomorphism numbers	56
5.3. What hom functions can express	62
5.4. Homomorphism and isomorphism	68
5.5. Independence of homomorphism functions	72
5.6. Characterizing homomorphism numbers	75
5.7. The structure of the homomorphism set	79
Chapter 6. Graph algebras and homomorphism functions	83
6.1. Algebras of quantum graphs	83
6.2. Reflection positivity	88

6.3. Contractors and connectors	94
6.4. Algebras for homomorphism functions	101
6.5. Computing parameters with finite connection rank	106
6.6. The polynomial method	108
Part 3. Limits of dense graph sequences	113
Chapter 7. Kernels and graphons	115
7.1. Kernels, graphons and stepfunctions	115
7.2. Generalizing homomorphisms	116
7.3. Weak isomorphism I	121
7.4. Sums and products	122
7.5. Kernel operators	124
Chapter 8. The cut distance	127
8.1. The cut distance of graphs	127
8.2. Cut norm and cut distance of kernels	131
8.3. Weak and L_1 -topologies	138
Chapter 9. Szemerédi partitions	141
9.1. Regularity Lemma for graphs	141
9.2. Regularity Lemma for kernels	144
9.3. Compactness of the graphon space	149
9.4. Fractional and integral overlays	151
9.5. Uniqueness of regularity partitions	154
Chapter 10. Sampling	157
10.1. W -random graphs	157
10.2. Sample concentration	158
10.3. Estimating the distance by sampling	160
10.4. The distance of a sample from the original	164
10.5. Counting Lemma	167
10.6. Inverse Counting Lemma	169
10.7. Weak isomorphism II	170
Chapter 11. Convergence of dense graph sequences	173
11.1. Sampling, homomorphism densities and cut distance	173
11.2. Random graphs as limit objects	174
11.3. The limit graphon	179
11.4. Proving convergence	185
11.5. Many disguises of graph limits	193
11.6. Convergence of spectra	194
11.7. Convergence in norm	196
11.8. First applications	197
Chapter 12. Convergence from the right	201
12.1. Homomorphisms to the right and multicuts	201
12.2. The overlay functional	205
12.3. Right-convergent graphon sequences	207
12.4. Right-convergent graph sequences	211

Chapter 13. On the structure of graphons	217
13.1. The general form of a graphon	217
13.2. Weak isomorphism III	220
13.3. Pure kernels	222
13.4. The topology of a graphon	225
13.5. Symmetries of graphons	234
Chapter 14. The space of graphons	239
14.1. Norms defined by graphs	239
14.2. Other norms on the kernel space	242
14.3. Closures of graph properties	247
14.4. Graphon varieties	250
14.5. Random graphons	256
14.6. Exponential random graph models	259
Chapter 15. Algorithms for large graphs and graphons	263
15.1. Parameter estimation	263
15.2. Distinguishing graph properties	266
15.3. Property testing	268
15.4. Computable structures	276
Chapter 16. Extremal theory of dense graphs	281
16.1. Nonnegativity of quantum graphs and reflection positivity	281
16.2. Variational calculus of graphons	283
16.3. Densities of complete graphs	285
16.4. The classical theory of extremal graphs	293
16.5. Local vs. global optima	294
16.6. Deciding inequalities between subgraph densities	299
16.7. Which graphs are extremal?	307
Chapter 17. Multigraphs and decorated graphs	317
17.1. Compact decorated graphs	318
17.2. Multigraphs with unbounded edge multiplicities	325
Part 4. Limits of bounded degree graphs	327
Chapter 18. Graphings	329
18.1. Borel graphs	329
18.2. Measure preserving graphs	332
18.3. Random rooted graphs	338
18.4. Subgraph densities in graphings	344
18.5. Local equivalence	346
18.6. Graphings and groups	349
Chapter 19. Convergence of bounded degree graphs	351
19.1. Local convergence and limit	351
19.2. Local-global convergence	360
Chapter 20. Right convergence of bounded degree graphs	367
20.1. Random homomorphisms to the right	367
20.2. Convergence from the right	375

Chapter 21. On the structure of graphings	383
21.1. Hyperfiniteness	383
21.2. Homogeneous decomposition	393
Chapter 22. Algorithms for bounded degree graphs	397
22.1. Estimable parameters	397
22.2. Testable properties	402
22.3. Computable structures	405
Part 5. Extensions: a brief survey	413
Chapter 23. Other combinatorial structures	415
23.1. Sparse (but not very sparse) graphs	415
23.2. Edge-coloring models	416
23.3. Hypergraphs	421
23.4. Categories	425
23.5. And more...	429
Appendix A. Appendix	433
A.1. Möbius functions	433
A.2. The Tutte polynomial	434
A.3. Some background in probability and measure theory	436
A.4. Moments and the moment problem	441
A.5. Ultraproduct and ultralimit	444
A.6. Vapnik–Chervonenkis dimension	445
A.7. Nonnegative polynomials	446
A.8. Categories	447
Bibliography	451
Author Index	465
Subject Index	469
Notation Index	473