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Regularity Theory for Elliptic PDE

One of the most basic mathematical questions in PDE is that of regularity. A classical example is Hilbert's XIXth problem, stated in 1900, which was solved by De Giorgi and Nash in the 1950's. The question of regularity has been a central line of research in elliptic PDE during the second half of the 20th century and has influenced many areas of mathematics linked one way or another with PDE.

This text aims to provide a self-contained introduction to the regularity theory for elliptic PDE, focusing on the main ideas rather than proving all results in their greatest generality. It can be seen as a bridge between an elementary PDE course and more advanced books.

The book starts with a short review of the Laplace operator and harmonic functions. The theory of Schauder estimates is developed next, but presented with various proofs of the results. Nonlinear elliptic PDE are covered in the following, both in the variational and non-variational setting and, finally, the obstacle problem is studied in detail, establishing the regularity of solutions and free boundaries.

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