

This book provides the most comprehensive treatment of the theoretical concepts and modelling techniques of quantitative risk management. Whether you are a financial risk analyst, actuary, regulator or student of quantitative finance, *Quantitative Risk Management* gives you the practical tools you need to solve real-world problems.

Describing the latest advances in the field, *Quantitative Risk Management* covers the methods for market, credit and operational risk modelling. It places standard industry approaches on a more formal footing and explores key concepts such as loss distributions, risk measures and risk aggregation and allocation principles. The book's methodology draws on diverse quantitative disciplines, from mathematical finance and statistics to econometrics and actuarial mathematics. A primary theme throughout is the need to satisfactorily address extreme outcomes and the dependence of key risk drivers. Proven in the classroom, the book also covers advanced topics like credit derivatives.

- Fully revised and expanded to reflect developments in the field since the financial crisis
- Features shorter chapters to facilitate teaching and learning
- Provides enhanced coverage of Solvency II and insurance risk management and extended treatment of credit risk, including counterparty credit risk and CDO pricing
- Includes a new chapter on market risk and new material on risk measures and risk aggregation

PRAISE FOR THE PREVIOUS EDITION:

"Quantitative Risk Management can be highly recommended to anyone looking for an excellent survey of the most important techniques and tools used in this rapidly growing field."

—**HOLGER DREES**, *Risk*

"Quantitative Risk Management is highly recommended for financial regulators. The statistical and mathematical tools facilitate a better understanding of the strengths and weaknesses of a useful range of advanced risk-management concepts and models, while the focus on aggregate risk enhances the publication's value to banking and insurance supervisors."

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"This book is a compendium of the statistical arrows that should be in any quantitative risk manager's quiver. It includes extensive discussion of dynamic volatility models, extreme value theory, copulas and credit risk. Academics, PhD students and quantitative practitioners will find many new and useful results in this important volume."

—**ROBERT F. ENGLE III**, 2003 Nobel Laureate in Economic Sciences, Michael Armellino Professor in the Management of Financial Services at New York University's Stern School of Business

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