

This book provides a detailed treatment of the various facets of modern Sturm–Liouville theory, including such topics as Weyl–Titchmarsh theory, classical, renormalized, and perturbative oscillation theory, boundary data maps, traces and determinants for Sturm–Liouville operators, strongly singular Sturm–Liouville differential operators, generalized boundary values, and Sturm–Liouville operators with distributional coefficients. To illustrate the theory, the book develops an array of examples from Floquet theory to short-range scattering theory, higher-order KdV trace relations, elliptic and algebro-geometric finite gap potentials, reflectionless potentials and the Sodin–Yuditskii class, as well as a detailed collection of singular examples, such as the Bessel, generalized Bessel, and Jacobi operators. A set of appendices contains background on the basics of linear operators and spectral theory in Hilbert spaces, Schatten–von Neumann classes of compact operators, self-adjoint extensions of symmetric operators, including the Friedrichs and Krein–von Neumann extensions, boundary triplets for ODEs, Krein-type resolvent formulas, sesquilinear forms, Nevanlinna–Herglotz functions, and Bessel functions.

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