

In the last sixty years, the use of the notion of category has led to a remarkable unification and simplification of mathematics. *Conceptual Mathematics* introduces this tool for the learning, development, and use of mathematics, to beginning students and also to practising mathematical scientists. This book provides a skeleton key that makes explicit some concepts and procedures that are common to all branches of pure and applied mathematics. The treatment does not presuppose knowledge of specific fields, but rather develops, from basic definitions, such elementary categories as discrete dynamical systems and directed graphs; the fundamental ideas are then illuminated by examples in these categories.

This second edition provides links with more advanced topics of possible study. In the new appendices and annotated bibliography the reader will find concise introductions to adjoint functors and geometrical structures, as well as sketches of relevant historical developments.

‘This text written by two experts in category theory and tried out carefully in courses at SUNY Buffalo, provides a simple and effective first course on conceptual mathematics.’

*American Mathematical Monthly*

‘Every mathematician should know the basic ideas and techniques explained in this book.’

*Monatshefte für Mathematik*

‘Category theory slices across the artificial boundaries dividing algebra, arithmetic, calculus, geometry, logic, topology. If you have students you wish to introduce to the subject, I suggest this delightfully elementary book. I have recommended this book to motivated high school students. I certainly suggest it for undergraduates. I even suggest it for the mathematician who needs a refresher on modern concepts.’

*National Association of Mathematicians Newsletter*

‘*Conceptual Mathematics* provides an excellent introductory account of categories for those who are starting from scratch. It treats material that will appear simple and familiar to many philosophers, but in an unfamiliar way.’

*Studies in History and Philosophy of Modern Physics*

**CAMBRIDGE**  
UNIVERSITY PRESS  
[www.cambridge.org](http://www.cambridge.org)

ISBN 978-0-521-71916-2



9 780521 719162 >





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