

Contents

<i>Preface</i>	<i>page</i>	xiii
<i>Acknowledgments</i>	<i>page</i>	xiv
1 Yes You <i>Can</i> Prove a Negative!		1
1.1 Introduction to Impossibility		1
1.2 What These Theorems and Proofs Have in Common		3
1.3 This Book Builds on These Improvements		4
1.4 Why Proofs?		5
1.5 OK, Proofs Are Important, but Why Another Book?		6
1.5.1 Because		6
1.5.2 In Contrast		7
1.5.3 Other Impossibilities		8
1.6 How to Read This Book		8
2 Bell's Impossibility Theorem(s)		10
2.1 Bell, Einstein, and All That		10
2.2 Is Impossibility Possible or Impossible?		13
2.2.1 Models, Proofs, and Experiments		14
2.2.1.1 Mermin's Model		14
2.2.1.2 What Is Predicted, and Observed, in an EPR Experiment?		16
2.2.1.3 Modeling Einsteinian Theories		17
2.2.2 Now Impossibility Is Possible		19
2.2.3 Bell's Theorem for the Index-card Model		20
2.2.4 Where Is the Inequality?		23
2.3 Why Is This Shocking and Profound?		23
2.3.1 The Backstory		24
2.3.2 A Fictional Example of Entanglement		25
2.3.2.1 Recapping, and Continuing		27

2.3.3	A Bit More about EPR Experiments	28
2.3.4	Einstein and the Unstated Assumptions	28
2.3.4.1	First Assumption: Physical Reality	29
2.3.4.2	Second Assumption: Locality	29
2.3.5	What Kind of Theory Did Einstein Insist On?	30
2.3.6	But How?	31
2.3.7	Now We Can See What Is so Shocking and Profound	33
2.4	Exercises	36
3	Enjoying Bell Magic: With Inequalities and Without	40
3.1	A Derivation of Bell's Central Inequality	40
3.1.1	A Fictional Town and Story	40
3.2	A Bell Inequality	42
3.2.1	Now We Prove It	42
3.2.2	Nothing Tricky in This Proof	43
3.2.3	Connecting Bell's Inequality to EPR	44
3.2.4	And Yet!	45
3.3	Why Is Bell's Inequality So Important?	46
3.3.1	Local Realism Again	46
3.4	Is There a More Insightful Explanation?	47
3.4.1	Try This One	47
3.5	Using Impossibility to Create Possibility	51
3.5.1	But How?	52
3.6	GHZ: An Even More Magical Experiment and Amazing Theorem (with an Easy, Elementary Proof)	53
3.6.1	Why This Matters	54
3.6.2	Bell's Theorem via the GHZ Experiment	54
3.6.2.1	Can an Index-card Model Agree with GHZ1 and GHZ2?	55
3.6.2.2	The Answer Is No!	55
3.6.3	From First Principles	57
3.6.4	Certainty and Experiment	59
3.6.5	Another Confession	61
3.7	Surprise Bonus: The Hardy Experiment and an (Almost) Trivial Proof of Bell's Theorem	62
3.7.1	The Experiment	62
3.8	What Impossibility Proves	64
3.8.1	Final Praise	65
3.9	Exercises	66

4	Arrow's (and Friends') Impossibility Theorems	78
4.1	Are Fair Elections Possible?	78
4.2	First, a Related Impossibility Theorem	81
4.2.1	Requirements for a Fair Single-Winner Election Mechanism	81
4.2.2	Now Back to Theorem 4.2.1	85
4.2.2.1	Proof of Theorem 4.2.1	85
4.2.3	Back to GS	92
4.2.3.1	Deceit-Immune Election Mechanisms	93
4.2.4	The Actual GS Impossibility Theorem	94
4.2.5	A Small Sermon about “Useless” Knowledge	97
4.3	Arrow's Impossibility Theorem	97
4.3.1	Arrow's Requirements for a Fair Election Mechanism	98
4.4	A Proof of Arrow's General Theorem	101
4.4.1	Act One: The Start of a Scenario	102
4.4.2	Act Two: Onward with Scenario-A	105
4.4.3	Back to Scenario-A	106
4.4.4	The Last Modifications	107
4.4.5	The Final Act: Another Renaming Argument	110
4.5	Exercises	112
5	Clustering and Impossibility	114
5.1	The Importance of Clustering	114
5.1.1	Clusters and Clustering	114
5.2	Clustering Axioms and Impossibility	119
5.2.1	Kleinberg's Axioms	120
5.2.2	More Realistic Axioms	124
5.3	Take-Home Lessons	126
5.3.1	Proof vs Practice	127
5.4	Exercises	128
6	A Gödel-ish Impossibility and Incompleteness Theorem	130
6.1	Introduction	130
6.1.1	In This Chapter	132
6.2	The First Key Idea: There are Non-computable Functions	134
6.3	What Is a Formal Proof System?	138
6.3.1	What Is a Formal Derivation?	139
6.3.2	Mechanical Generation and Checking of Formal Derivations	140
6.4	Back to Gödel and Incompleteness	141

6.4.1	What Is Truth?	142
6.4.2	Finally, Our Variant of Gödel's Theorem	143
6.5	Alternate Statements of (Our Variant of) Gödel's First Incompleteness Theorem	144
6.5.1	Consistency and Soundness	145
6.6	What Have We Learned, and What More Do We Want?	146
6.6.1	We Are Not There Yet	147
6.6.2	And What Else?	148
6.7	Exercises	148
7	Turing Undecidability and Incompleteness	149
7.1	Undecidable Decision Problems	149
7.2	The Halting Problem Is Undecidable	150
7.2.1	Not Halting Is (Usually) Not Good	151
7.2.2	The Self-Halting Problem	153
7.2.2.1	The Analysis of the Self-Halting Problem	153
7.2.2.2	And Now for the Kicker	157
7.2.3	Turing's Proof in Rhyme	157
7.2.4	Is the Halting Problem Special?	158
7.2.4.1	Another Important Undecidable Problem	159
7.3	Using the Halting Problem to Establish Incompleteness	160
7.3.1	A Dramatic Turn	160
7.3.2	Isn't This <i>Déjà vu</i> All Over Again?	163
7.4	Advanced Topic: From Soundness to Consistency	165
7.4.1	On to Consistency	167
7.5	<i>P</i> vs. <i>NP</i> : In the <i>Spirit</i> of Impossibility	170
7.6	Exercises	171
8	Even More Devastating: Chaitin's Incompleteness Theorem	174
8.1	Perplexing and Devastating	174
8.2	Strings and Compression	174
8.3	Chaitin's Incompleteness Theorem	176
8.3.1	Statement of Chaitin's Theorem	176
8.4	$K_{\mathcal{L}}$ Is Not Computable	177
8.5	Now Back to Chaitin's Theorem and Proof	178
8.5.1	Preparatory point 2: Program P_{gc}	179

8.5.1.1	Preparatory point 3: Programs P_g and P_c Alternate Executions	180
8.5.1.2	How Big Is Program P_{gc} ?	181
8.5.1.3	Preparatory point 4: How to choose $u_{\mathcal{L}}$	182
8.6	And Now the Proof	183
8.6.1	Shocking?	183
8.6.2	Replacing “Sound” with “Consistent”	184
8.6.2.1	What Is Missing?	185
8.7	What Is So “Devastating” about Chaitin’s Theorem	186
8.7.1	OK, That Was Fun, but Seriously Now	188
9	Gödel (For Real, This Time)	190
9.1	Background	191
9.2	The Elements of \mathcal{L}_A , a Language of Arithmetic	192
9.2.1	The Alphabet of \mathcal{L}_A	192
9.2.2	Examples of Expressions in \mathcal{L}_A	194
9.2.3	Predicates	196
9.2.4	Formalizing the Concept of “Expressing”	198
9.2.5	Recapping	199
9.2.6	An <i>Abstract</i> Formal System Π	199
9.3	Gödel Numbers and the Diagonal Function	199
9.3.1	First Key Tool: Gödel Numbers	200
9.3.2	Second Key Tool: The Diagonal Function $d()$	202
9.3.2.1	The Diagonal Function in a Nutshell	203
9.3.3	One Final Definition Before the Main Acts	204
9.4	A Diagonal Lemma	205
9.4.1	Now We Prove the Diagonal Lemma	206
9.5	Now, an Actual Gödel Incompleteness Theorem	206
9.5.1	And, the Proof	207
9.5.2	A Gödel Sentence Was Used	208
9.5.2.1	The Heart of It	208
9.5.3	Expressing $d^{-1}(\tilde{\mathcal{P}})$: Another Three-Step Plan	209
9.5.3.1	Computation and Logic	210
9.5.3.2	Recapping	214
9.6	Gödel’s Second Incompleteness Theorem	214
9.6.1	Informally	215
9.6.2	It’s an Inside Job	215
9.6.3	Say It in \mathcal{L}_A	217
9.6.4	Proof of the Second Theorem: Gödel Sketcher Is Back	218

9.6.5	Another Way of Explaining the Proof	219
9.6.6	What Does “Rich-Enough” Really Mean?	220
9.7	Misconceptions about Gödel’s Theorems	221
9.7.1	Misconceptions about the First Theorem	221
9.7.2	Misconceptions about the Second Theorem	223
9.7.2.1	A Perplexing Technical Misconception	224
9.7.3	Misconceptions about the Causes of Incompleteness	227
9.7.3.1	Gödel Numbering Based on Prime Powers	227
9.7.3.2	Explicit Self-Reference	229
9.7.3.3	Twisty Gödel Sentences and Paradoxes	230
9.7.4	Not a Misconception, but a Difference of Opinion	231
9.8	Exercises	233
<i>Appendix Computer Programs Are Text</i>		234
<i>Bibliography</i>		240
<i>Index</i>		246