

Fundamental concepts of phase transitions, such as order parameters, spontaneous symmetry breaking, scaling transformations, conformal symmetry and anomalous dimensions, have deeply changed the modern vision of many areas of physics, leading to remarkable developments in statistical mechanics, elementary particle theory, condensed matter physics and string theory. This self-contained book provides a thorough introduction to the fascinating world of phase transitions and frontier topics of exactly solved models in statistical mechanics and quantum field theory, such as renormalization groups, conformal models, quantum integrable systems, duality, elastic S-matrices, thermodynamic Bethe ansatz and form factor theory. The clear discussion of physical principles is accompanied by a detailed analysis of several branches of mathematics distinguished for their elegance and beauty, including infinite dimensional algebras, conformal mappings, integral equations and modular functions.

Besides advanced research themes, the book also covers many basic topics in statistical mechanics, quantum field theory and theoretical physics. Each argument is discussed in great detail while providing overall coherent understanding of physical phenomena. Mathematical background is made available in supplements at the end of each chapter, when appropriate. The chapters include problems of different levels of difficulty. Advanced undergraduate and graduate students will find this book a rich and challenging source for improving their skills and for attaining a comprehensive understanding of the many facets of the subject.

Giuseppe Mussardo is Professor of Theoretical Physics at the International School for Advanced Studies (ISAS) in Trieste.

#### REVIEWS OF THE FIRST EDITION

'The book is well suited to provide access into this fascinating field of research and at the same time leads its readers all the way to the forefront of present research. Such a book will provide a solid basis as a textbook for an advanced course in statistical physics, giving the lecturer an ample choice of topics supplemented by problem sets and references to the original literature.'

Holger Frahm, Leibniz University, Hannover

'I am very impressed with the contents of this book—it is certainly needed. The author is a good writer and can explain things well. From the scientific point of view the quality is outstanding.'

Alexei Tsvelik, Brookhaven National Laboratory

'The author is an excellent physicist who has contributed very significantly to the field, and has always shown a passion for pedagogy. He is one of a handful of people who look beyond formal theory and think in terms of physics, always trying to push the boundaries of our knowledge. I am sure this book will become a most useful and successful text for graduate students and researchers.'

Hubert Saleur, University of Southern California

Cover image: A field configuration in the continuum limit of lattice statistical models

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