

# Rays, Waves, and Scattering

John A. Adam

This one-of-a-kind book presents many of the mathematical concepts, structures, and techniques used in the study of rays, waves, and scattering. Panoramic in scope, it includes discussions of how ocean waves are refracted around islands and underwater ridges, how seismic waves are refracted in the earth's interior, how atmospheric waves are scattered by mountains and ridges, how the scattering of light waves produces the blue sky, and meteorological phenomena such as rainbows and coronas.

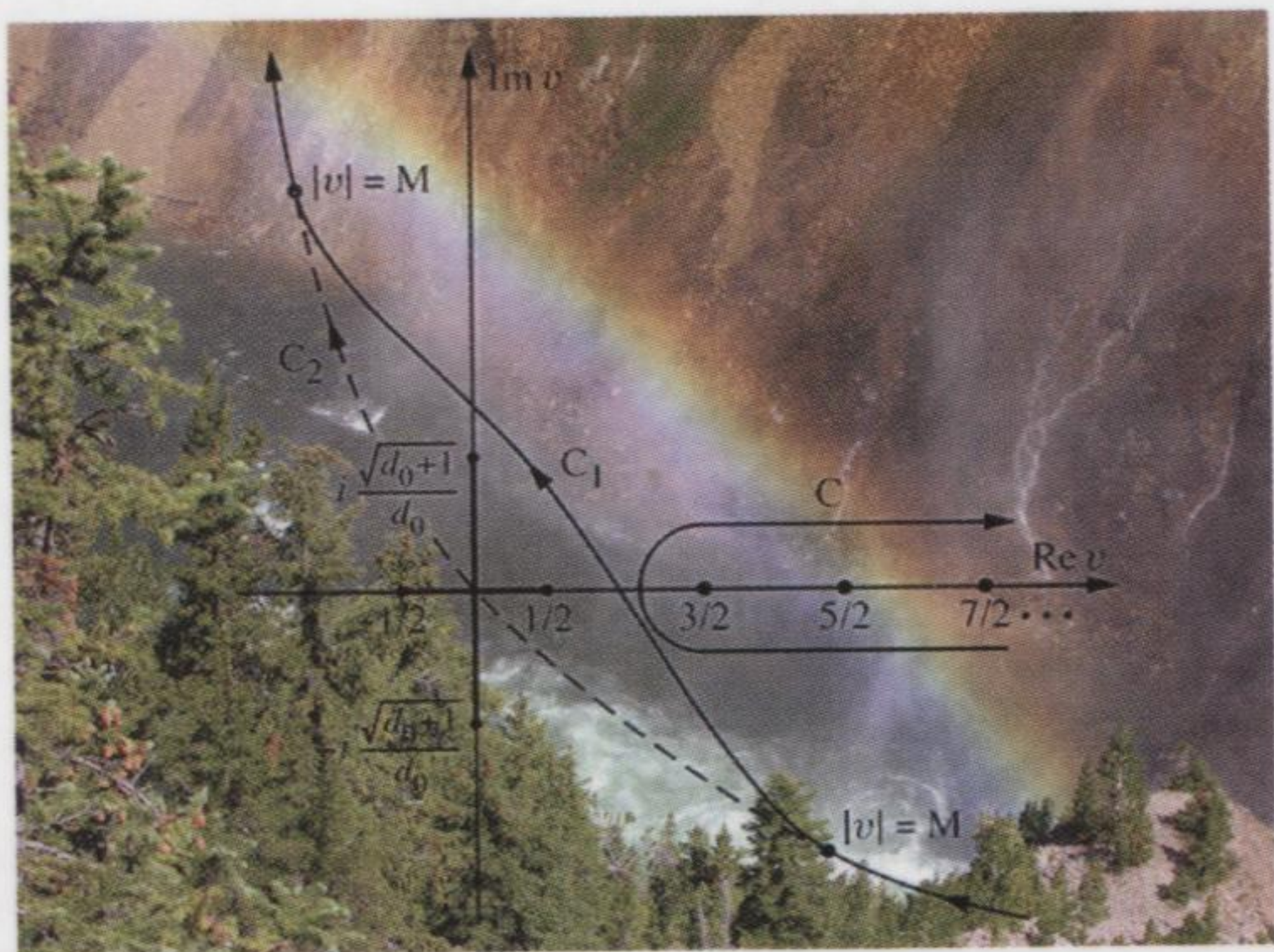
*Rays, Waves, and Scattering* is a valuable resource for practitioners, graduate students, and advanced undergraduates in applied mathematics, theoretical physics, and engineering. Bridging the gap between advanced treatments of the subject written for specialists and less mathematical books aimed at beginners, this unique mathematical compendium features problems and exercises throughout that are geared to various levels of sophistication, covering everything from Ptolemy's theorem to Airy integrals (as well as more technical material), and several informative appendixes.

- Provides a panoramic look at wave motion in many different contexts
- Features problems and exercises throughout
- Includes numerous appendixes, some on topics not often covered
- An ideal reference book for practitioners
- Can also serve as a supplemental text in classical applied mathematics, particularly wave theory and mathematical methods in physics and engineering
- Accessible to anyone with a strong background in ordinary differential equations, partial differential equations, and functions of a complex variable

"This is a significant contribution to the literature on wave theory, one that blends the mathematics and physics in just the right way. All derivations are given in full so that the work is genuinely a students' book, and Adam has chosen only the most interesting parts of the subject, especially those masterpieces of elegance that draw people to advanced physics in the first place."


—C. J. Chapman, Keele University

**John A. Adam** is professor of mathematics at Old Dominion University. His books include *X and the City: Modeling Aspects of Urban Life*, *Mathematics in Nature: Modeling Patterns in the Natural World*, and *Guesstimation: Solving the World's Problems on the Back of a Cocktail Napkin* (all Princeton).



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