

A number of recent developments in the one interval and two interval theory of Sturm-Liouville problems are discussed. These include operators with periodic coefficients, self-adjoint operators with finite spectrum and their equivalent matrix problems, inverse theory of operators with finite spectrum, eigenvalues below the essential spectrum and eigenvalue problems with parameter dependent boundary conditions. Moreover, self-adjoint operators with discontinuous boundary conditions and their Green' function, such conditions are known by various names, including transmission conditions, interface conditions, multi-point conditions, etc., are discussed as well. Also, an algorithm is developed which can be used, with appropriate software, to compute the eigenvalues of regular and singular Sturm-Liouville problems with coupled boundary conditions using the Prüfer transformation on families of problems with separated boundary conditions.

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Contents

Introduction — V

Part I: One-interval problems

1 Classical regular self-adjoint problems — 3

- 1.1 Introduction — 3
- 1.2 Self-adjoint operators in Hilbert space — 5
- 1.3 Canonical forms of self-adjoint boundary conditions — 6
- 1.4 Existence of eigenvalues — 9
- 1.5 Continuity of eigenvalues — 11
- 1.6 Differentiability of eigenvalues — 13
- 1.7 Eigenvalue inequalities — 15
- 1.8 Monotonicity and multiplicity of eigenvalues — 19
- 1.9 The Prüfer transformation and separated boundary conditions — 20
- 1.10 A Prüfer characterization for real coupled conditions — 24
- 1.11 Another family of separated boundary conditions — 27
- 1.12 Proof of the algorithm — 31
- 1.13 Comments — 32

2 Periodic coefficients — 35

- 2.1 Eigenvalues of periodic, semiperiodic, and complex boundary conditions — 35
- 2.2 General eigenvalue inequalities — 37
- 2.3 Structure of solutions — 39
- 2.4 Eigenvalues on one interval — 42
- 2.5 Eigenvalues on different intervals — 44
- 2.6 Eigenvalues of periodic, semiperiodic, and complex boundary conditions — 51
- 2.7 Eigenvalue equalities from different intervals — 52
- 2.8 Construction of the one-to-one correspondence — 56
- 2.9 Examples of the one-to-one correspondence — 59
- 2.10 Spectrum of the minimal operator — 63
- 2.11 Comments — 64

3 Extensions of the classical problem — 67

- 3.1 Introduction — 67
- 3.2 The leading coefficient changes sign — 68

3.3	Complex coefficients — 69
3.4	The weight function changes sign — 73
3.5	Nonnegative leading coefficient and weight function — 74
3.6	Comments — 78
4	Finite spectrum — 79
4.1	Introduction — 79
4.2	Matrix representations of Sturm–Liouville problems — 79
4.3	Sturm–Liouville representations of matrix eigenvalue problems — 86
4.4	The study of Jacobi and cyclic Jacobi matrix eigenvalue problems using Sturm–Liouville theory — 89
4.4.1	Main results — 90
4.5	Comments — 93
5	Inverse Sturm–Liouville problems with finite spectrum — 95
5.1	Introduction — 95
5.2	Main results — 96
5.3	Inverse matrix eigenvalue problems with a weight function — 99
5.4	Proofs of the main results — 103
5.5	Comments on the inverse theories for finite and infinite spectra — 107
6	Eigenvalues below the essential spectrum — 109
6.1	Introduction — 109
6.2	The Lagrange form and maximal and minimal domains — 109
6.3	Summary of spectral properties — 112
6.4	The LPNO case — 115
6.5	The general LP case — 119
6.6	Proofs of theorems in Section 6.4 — 121
6.7	Proofs of theorems in Section 6.5 — 126
6.8	Comments — 129
7	Spectral parameter in the boundary conditions — 131
7.1	Introduction — 131
7.2	Construction of operators — 131
7.2.1	The classical minimal and maximal operators in H_1 — 132
7.2.2	Construction of operators in H — 133
7.2.3	Inherited boundary conditions and induced restriction operators — 137
7.3	Spectral properties — 139
7.3.1	Assume that b is LC in H_1 — 139
7.3.2	Assume that b is LP in H_1 — 141
7.4	Approximation of eigenvalues — 142
7.4.1	The case where b is limit circle — 144

7.4.2	The case where b is limit point — 146
7.5	Examples — 148
7.6	Comments — 149

Part II: Two-interval problems

8	Discontinuous boundary conditions — 153
8.1	Introduction — 153
8.2	The one-interval theory — 154
8.3	The two-interval theory — 158
8.3.1	Regular endpoints — 166
8.3.2	Singular endpoints — 168
8.4	Transmission and interface conditions — 171
8.5	Comments — 179
9	The Green's and characteristic functions — 181
9.1	Introduction — 181
9.2	The characteristic function — 181
9.3	The Green's function — 184
9.4	Examples — 188
9.5	Comments — 191
10	The Legendre equation and its operators — 193
10.1	Introduction — 193
10.2	General properties — 194
10.3	Regular Legendre equations — 200
10.4	Self-adjoint operators in $L^2(-1, 1)$ — 205
10.4.1	Eigenvalue properties — 210
10.5	The maximal and Friedrichs domains — 213
10.6	The Legendre Green's function — 215
10.7	Operators on the interval $(1, +\infty)$ — 221
10.8	The Legendre operators on the whole line — 225
10.8.1	A self-adjoint Legendre operator on the whole real line — 230
10.9	Singular transmission and interface conditions for the Legendre equation — 231
10.10	Comments — 235
A	Notation — 237
B	Open problems — 239

Bibliography — 241

Index — 247