

Analytic Combinatorics

Analytic Combinatorics aims to enable precise quantitative predictions of the properties of large combinatorial structures. The theory has emerged over recent decades as essential both for the analysis of algorithms and for the study of scientific models in many disciplines, including probability theory, statistical physics, computational biology and information theory. With a careful combination of symbolic enumeration methods and complex analysis, drawing heavily on generating functions, results of sweeping generality emerge that can be applied in particular to fundamental structures such as permutations, sequences, strings, walks, paths, trees, graphs and maps.

This account is the definitive treatment of the topic. In order to make it self-contained, the authors provide full coverage of the underlying mathematics and give a thorough treatment of both classical and modern applications of the theory. The text is complemented with exercises, examples, appendices and notes throughout the book to aid understanding. The book can be used as a reference for researchers, as a textbook for an advanced undergraduate or a graduate course on the subject, or for self-study.

- **Comprehensive:** generous notes, appendices, examples and exercises, as well as the inclusion of proofs of fundamental results
- **Unified:** ties together classical mathematics and modern applications
- **Cutting edge:** first book with extensive coverage of analytic methods needed to analyse large combinatorial configurations

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AN INVITATION TO ANALYTIC COMBINATORICS

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