

Contents

Foreword, by Chris Isham	page	xvii
Preface	page	xix
Notation and conventions	page	xxiii
Introduction: Defining quantum gravity	page	1
Why quantum gravity in the twenty-first century?	page	1
The role of background independence	page	8
Approaches to quantum gravity	page	11
Motivation for canonical quantum general relativity	page	23
Outline of the book	page	25

I CLASSICAL FOUNDATIONS, INTERPRETATION AND THE CANONICAL QUANTISATION PROGRAMME

1 Classical Hamiltonian formulation of General Relativity	page	39
1.1 The ADM action	page	39
1.2 Legendre transform and Dirac analysis of constraints	page	46
1.3 Geometrical interpretation of the gauge transformations	page	50
1.4 Relation between the four-dimensional diffeomorphism group and the transformations generated by the constraints	page	56
1.5 Boundary conditions, gauge transformations and symmetries	page	60
1.5.1 Boundary conditions	page	60
1.5.2 Symmetries and gauge transformations	page	65
2 The problem of time, locality and the interpretation of quantum mechanics	page	74
2.1 The classical problem of time: Dirac observables	page	75
2.2 Partial and complete observables for general constrained systems	page	81
2.2.1 Partial and weak complete observables	page	82
2.2.2 Poisson algebra of Dirac observables	page	85
2.2.3 Evolving constants	page	89
2.2.4 Reduced phase space quantisation of the algebra of Dirac observables and unitary implementation of the multi-fingered time evolution	page	90
2.3 Recovery of locality in General Relativity	page	93

2.4	Quantum problem of time: physical inner product and interpretation of quantum mechanics	95
2.4.1	Physical inner product	95
2.4.2	Interpretation of quantum mechanics	98
3	The programme of canonical quantisation	107
3.1	The programme	108
4	The new canonical variables of Ashtekar for General Relativity	118
4.1	Historical overview	118
4.2	Derivation of Ashtekar's variables	123
4.2.1	Extension of the ADM phase space	123
4.2.2	Canonical transformation on the extended phase space	126

II FOUNDATIONS OF MODERN CANONICAL QUANTUM GENERAL RELATIVITY

5	Introduction	141
5.1	Outline and historical overview	141
6	Step I: the holonomy–flux algebra \mathfrak{P}	157
6.1	Motivation for the choice of \mathfrak{P}	157
6.2	Definition of \mathfrak{P} : (1) Paths, connections, holonomies and cylindrical functions	162
6.2.1	Semianalytic paths and holonomies	162
6.2.2	A natural topology on the space of generalised connections	168
6.2.3	Gauge invariance: distributional gauge transformations	175
6.2.4	The C^* algebraic viewpoint and cylindrical functions	183
6.3	Definition of \mathfrak{P} : (2) surfaces, electric fields, fluxes and vector fields	191
6.4	Definition of \mathfrak{P} : (3) regularisation of the holonomy–flux Poisson algebra	194
6.5	Definition of \mathfrak{P} : (4) Lie algebra of cylindrical functions and flux vector fields	202
7	Step II: quantum $*$-algebra \mathfrak{A}	206
7.1	Definition of \mathfrak{A}	206
7.2	(Generalised) bundle automorphisms of \mathfrak{A}	209
8	Step III: representation theory of \mathfrak{A}	212
8.1	General considerations	212
8.2	Uniqueness proof: (1) existence	219
8.2.1	Regular Borel measures on the projective limit: the uniform measure	220
8.2.2	Functional calculus on a projective limit	226

8.2.3	+ Density and support properties of $\mathcal{A}, \mathcal{A}/\mathcal{G}$ with respect to $\overline{\mathcal{A}}, \overline{\mathcal{A}/\mathcal{G}}$	435
8.2.4	Spin-network functions and loop representation	233
8.2.5	Gauge and diffeomorphism invariance of μ_0	237
8.2.6	+ Ergodicity of μ_0 with respect to spatial diffeomorphisms	242
8.2.7	Essential self-adjointness of electric flux momentum operators	245
8.3	Uniqueness proof: (2) uniqueness	246
8.4	Uniqueness proof: (3) irreducibility	247
9	Step IV: (1) implementation and solution of the kinematical constraints	252
9.1	Implementation of the Gauß constraint	264
9.1.1	Derivation of the Gauß constraint operator	264
9.1.2	Complete solution of the Gauß constraint	266
9.2	Implementation of the spatial diffeomorphism constraint	269
9.2.1	Derivation of the spatial diffeomorphism constraint operator	269
9.2.2	General solution of the spatial diffeomorphism constraint	271
10	Step IV: (2) implementation and solution of the Hamiltonian constraint	279
10.1	Outline of the construction	279
10.2	Heuristic explanation for UV finiteness due to background independence	282
10.3	Derivation of the Hamiltonian constraint operator	286
10.4	Mathematical definition of the Hamiltonian constraint operator	291
10.4.1	Concrete implementation	291
10.4.2	Operator limits	296
10.4.3	Commutator algebra	300
10.4.4	The quantum Dirac algebra	309
10.5	The kernel of the Wheeler–DeWitt constraint operator	311
10.6	The Master Constraint Programme	317
10.6.1	Motivation for the Master Constraint Programme in General Relativity	317
10.6.2	Definition of the Master Constraint	320
10.6.3	Physical inner product and Dirac observables	326
10.6.4	Extended Master Constraint	329
10.6.5	Algebraic Quantum Gravity (AQG)	331
10.7	+ Further related results	334
10.7.1	The Wick transform	334
10.7.2	Testing the new regularisation technique by models of quantum gravity	340

10.7.3	Quantum Poincaré algebra	341
10.7.4	Vasiliev invariants and discrete quantum gravity	344
11	Step V: semiclassical analysis	345
11.1	+ Weaves	349
11.2	Coherent states	353
11.2.1	Semiclassical states and coherent states	354
11.2.2	Construction principle: the complexifier method	356
11.2.3	Complexifier coherent states for diffeomorphism-invariant theories of connections	362
11.2.4	Concrete example of complexifier	367
11.2.5	Semiclassical limit of loop quantum gravity: graph-changing operators, shadows and diffeomorphism-invariant coherent states	376
11.2.6	+ The infinite tensor product extension	385
11.3	Graviton and photon Fock states from $L_2(\overline{\mathcal{A}}, d\mu_0)$	390
III PHYSICAL APPLICATIONS		411
12	Extension to standard matter	399
12.1	The classical standard model coupled to gravity	400
12.1.1	Fermionic and Einstein contribution	401
12.1.2	Yang–Mills and Higgs contribution	405
12.2	Kinematical Hilbert spaces for diffeomorphism-invariant theories of fermion and Higgs fields	406
12.2.1	Fermionic sector	406
12.2.2	Higgs sector	411
12.2.3	Gauge and diffeomorphism-invariant subspace	417
12.3	Quantisation of matter Hamiltonian constraints	418
12.3.1	Quantisation of Einstein–Yang–Mills theory	419
12.3.2	Fermionic sector	422
12.3.3	Higgs sector	425
12.3.4	A general quantisation scheme	429
13	Kinematical geometrical operators	431
13.1	Derivation of the area operator	432
13.2	Properties of the area operator	434
13.3	Derivation of the volume operator	438
13.4	Properties of the volume operator	447
13.4.1	Cylindrical consistency	447
13.4.2	Symmetry, positivity and self-adjointness	448
13.4.3	Discreteness and anomaly-freeness	448
13.4.4	Matrix elements	449
13.5	Uniqueness of the volume operator, consistency with the flux operator and pseudo-two-forms	453

13.6	Spatially diffeomorphism-invariant volume operator	455
14	Spin foam models	458
14.1	Heuristic motivation from the canonical framework	458
14.2	Spin foam models from BF theory	462
14.3	The Barrett–Crane model	466
14.3.1	Plebanski action and simplicity constraints	466
14.3.2	Discretisation theory	472
14.3.3	Discretisation and quantisation of BF theory	476
14.3.4	Imposing the simplicity constraints	482
14.3.5	Summary of the status of the Barrett–Crane model	494
14.4	Triangulation dependence and group field theory	495
14.5	Discussion	502
15	Quantum black hole physics	511
15.1	Classical preparations	514
15.1.1	Null geodesic congruences	514
15.1.2	Event horizons, trapped surfaces and apparent horizons	517
15.1.3	Trapping, dynamical, non-expanding and (weakly) isolated horizons	519
15.1.4	Spherically symmetric isolated horizons	526
15.1.5	Boundary symplectic structure for SSIHs	535
15.2	Quantisation of the surface degrees of freedom	540
15.2.1	Quantum U(1) Chern–Simons theory with punctures	541
15.3	Implementing the quantum boundary condition	546
15.4	Implementation of the quantum constraints	548
15.4.1	Remaining U(1) gauge transformations	549
15.4.2	Remaining surface diffeomorphism transformations	550
15.4.3	Final physical Hilbert space	550
15.5	Entropy counting	550
15.6	Discussion	557
16	Applications to particle physics and quantum cosmology	562
16.1	Quantum gauge fixing	562
16.2	Loop Quantum Cosmology	563
17	Loop Quantum Gravity phenomenology	572
IV MATHEMATICAL TOOLS AND THEIR CONNECTION TO PHYSICS		
18	Tools from general topology	577
18.1	Generalities	577
18.2	Specific results	581

19 Differential, Riemannian, symplectic and complex geometry	585
19.1 Differential geometry	585
19.1.1 Manifolds	585
19.1.2 Passive and active diffeomorphisms	587
19.1.3 Differential calculus	590
19.2 Riemannian geometry	606
19.3 Symplectic manifolds	614
19.3.1 Symplectic geometry	614
19.3.2 Symplectic reduction	616
19.3.3 Symplectic group actions	621
19.4 Complex, Hermitian and Kähler manifolds	623
20 Semianalytic category	627
20.1 Semianalytic structures on \mathbb{R}^n	627
20.2 Semianalytic manifolds and submanifolds	631
21 Elements of fibre bundle theory	634
21.1 General fibre bundles and principal fibre bundles	634
21.2 Connections on principal fibre bundles	636
22 Holonomies on non-trivial fibre bundles	644
22.1 The groupoid of equivariant maps	644
22.2 Holonomies and transition functions	647
23 Geometric quantisation	652
23.1 Prequantisation	652
23.2 Polarisation	662
23.3 Quantisation	668
24 The Dirac algorithm for field theories with constraints	671
24.1 The Dirac algorithm	671
24.2 First- and second-class constraints and the Dirac bracket	674
25 Tools from measure theory	680
25.1 Generalities and the Riesz–Markov theorem	680
25.2 Measure theory and ergodicity	687
26 Key results from functional analysis	689
26.1 Metric spaces and normed spaces	689
26.2 Hilbert spaces	691
26.3 Banach spaces	693
26.4 Topological spaces	694
26.5 Locally convex spaces	694
26.6 Bounded operators	695
26.7 Unbounded operators	697

26.8	Quadratic forms	699
27	Elementary introduction to Gel'fand theory for Abelian C^*-algebras	701
27.1	Banach algebras and their spectra	701
27.2	The Gel'fand transform and the Gel'fand isomorphism	709
28	Bohr compactification of the real line	713
28.1	Definition and properties	713
28.2	Analogy with loop quantum gravity	715
29	Operator $*$-algebras and spectral theorem	719
29.1	Operator $*$ -algebras, representations and GNS construction	719
29.2	Spectral theorem, spectral measures, projection valued measures, functional calculus	723
30	Refined algebraic quantisation (RAQ) and direct integral decomposition (DID)	729
30.1	RAQ	729
30.2	Master Constraint Programme (MCP) and DID	735
31	Basics of harmonic analysis on compact Lie groups	746
31.1	Representations and Haar measures	746
31.2	The Peter and Weyl theorem	752
32	Spin-network functions for $SU(2)$	755
32.1	Basics of the representation theory of $SU(2)$	755
32.2	Spin-network functions and recoupling theory	757
32.3	Action of holonomy operators on spin-network functions	762
32.4	Examples of coherent state calculations	765
33	+ Functional analytic description of classical connection dynamics	770
33.1	Infinite-dimensional (symplectic) manifolds	770
<i>References</i>		775
<i>Index</i>		809