

Advanced Calculus is intended as a text for courses that furnish the backbone of the student's undergraduate education in mathematical analysis. The goal is to rigorously present the fundamental concepts within the context of illuminating examples and stimulating exercises. This book is self-contained and starts with the creation of basic tools using the completeness axiom. The continuity, differentiability, integrability, and power series representation properties of functions of a single variable are established. The next few chapters describe the topological and metric properties of Euclidean space. These are the basis of a rigorous treatment of differential calculus (including the Implicit Function Theorem and Lagrange Multipliers) for mappings between Euclidean spaces and integration for functions of several real variables.

Special attention has been paid to the motivation for proofs. Selected topics, such as the Picard Existence Theorem for differential equations, have been included in such a way that selections may be made while preserving a fluid presentation of the essential material.

Supplemented with numerous exercises, *Advanced Calculus* is a perfect book for undergraduate students of analysis.

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mathematician and educator, Paul J. Sally, Jr.

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