

“This book accomplishes the impossible task: It explains to a mathematician, in a language that a mathematician can understand, what is meant by a quantum field theory from a physicist’s point of view. A great book by a great mathematician.”

– Sourav Chatterjee, Stanford University

“Talagrand has done an admirable job of making the difficult subject of quantum field theory as concrete and understandable as possible. The book progresses slowly and carefully but still covers an enormous amount of material. Talagrand has made every effort to assist the reader on a rewarding journey through the world of quantum fields.”

– Brian Hall, University of Notre Dame

“A presentation of the fundamental ideas of quantum field theory in a manner that is both accessible and mathematically accurate seems like an impossible dream. Well, not anymore! This book goes from basic notions to advanced topics with patience and care. It is an absolute delight to anyone looking for a friendly introduction to the beauty of quantum field theory and its mysteries.”

– Shahar Mendelson, Australian National University

“I have been motivated to try and learn about quantum field theories for some time but struggled to find a presentation in a language that I as a mathematician could understand. This book was perfect for me: I was able to make progress without any initial preparation and felt very comfortable and reassured by the style of exposition.”

– Ellen Powell, Durham University

“Michel Talagrand takes a decidedly elementary approach to answering the question in the title of his book, assuming little more than basic analysis. In addition to learning what quantum field theory is, the reader will encounter beautiful mathematics that is hard to find anywhere else in such clear pedagogical form. It is sure to remain a reference for many decades.”

– Philippe Sosoe, Cornell University

Cover image: The three shapes of diagrams (Fig. 13.15 in the book). Courtesy of the author.

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