

Handbook of Geometry and Topology of Singularities VII

This is the seventh volume of the Handbook of Geometry and Topology of Singularities, a series that aims to provide an accessible account of the state of the art of the subject, its frontiers, and its interactions with other areas of research.

This volume consists of fourteen chapters that provide an in-depth and reader-friendly introduction to various important aspects of singularity theory. The volume begins with an outstanding exposition on Jim Damon's contributions to singularity theory and its applications. Jim passed away in 2022 and he was one of the greatest mathematicians of recent times, having made remarkable contributions to singularity theory and its applications, mostly to medical image computing.

The next chapter focuses on the singularities of real functions and their bifurcation sets. Then, we look at the perturbation theory of polynomials and linear operators, complex analytic frontal singularities, the global singularity theory of differentiable maps, and the singularities of holomorphic functions from a global point of view.

The volume continues with an overview of new tools in singularity theory that spring from symplectic geometry and Floer-type homology theories; then, it looks at the derivation of Lie algebras of isolated singularities and the three-dimensional rational isolated complete intersection singularities, as well as recent developments in algebraic K-stability and the stable degeneration conjecture.

This volume also contains an interesting survey on V -filtrations, a theory began by Malgrange and Kashiwara that can be used to study nearby and vanishing cycle functors and introduced by Deligne. Then, we present a panoramic view of the Hodge, toric, and motivic methods in the study of Milnor fibers in singularity theory, both from local and global points of view.

The Monodromy conjecture is also explained; this is a longstanding open problem in singularity theory that lies at the crossroads of number theory, algebra, analysis, geometry, and topology. This volume closes with recent developments in the study of the algebraic complexity of optimization problems in applied algebraic geometry and algebraic statistics.

The book is addressed to graduate students and newcomers to the theory, as well as to specialists who can use it as a guidebook.

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