

## Invited lecture courses

**Bernhard Ruf:**

Exponential growth in 2-d . . . . . 1

**Armin Schikorra:**

Topological obstructions for Sobolev spaces . . . . . 61

**Nageswari Shanmugalingam:**

Short course on first order analysis in nonsmooth spaces  
using upper gradients . . . . . 99

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