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1.1 Financial Market

This book deals with computational finance problems with a focus on the analysis of numerical algorithms for pricing financial derivatives. There are two broad groups of numerical methods for derivative pricing: stochastic methods using random simulations and deterministic methods based on numerical solutions to partial differential equations of the Kolmogorov type. We cover both approaches in this book, emphasizing the mathematical backgrounds of the discussed algorithms.

We begin this chapter by reviewing basic information about the mathematical models of financial markets. This is a collection of definitions and facts. The purpose of this introductory part is to establish the terminology used in the book, and it cannot be considered a survey of mathematical modeling in finance. Readers interested in deeper insights into the mathematical modeling of financial markets should consult books by Eberlein and Kallsen [24], Hull [40], Joshi and Yor and Chesney [45], Karatzas and Shreve [40], Lamberon and Lapeyre [55], Shiryaev [81], or Shreve [82].

We consider a financial market in continuous time. When modeling a real market, we typically assume that the time horizon is finite. However, the probabilistic model is valid for any time interval. Thus, in this collection of facts and definitions, we assume that time t belongs to an interval \mathbb{T} , where $\mathbb{T} = [0, \infty)$ or $\mathbb{T} = [0, T]$ with $T < \infty$. Uncertainty in the market is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The probability space is assumed to be complete: if $B \subset A$